Functiona Analysis_v5

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Summary of the Course: Functional Analysis (Math–412)

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Preface

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Chapter 1

Metric Spases

Definition 1 *Let X be a non-empty set ($X \neq \emptyset$) and d a real-valued function defined on $X \times X$*

$$d: X \times X \to \mathbb{R}$$
$$(a, b) \to d(a, b)$$

such that for $a,\ b \in X$:
(i) $d(a, b) \geq 0$ and $d(a, b) = 0$ if and only if $a = b$.
(ii) $d(a, b) = d(b, a)$.
(iii) $d(a, c) \leq d(a, b) + d(b, c)$ (the triangle inequality) for $a,\ b$ and c in X.
*Then, d is said to be a **metric** on X, (X, d) is called a **metric space**, and $d(a, b)$ is referred to as the **distance** between a and b.*

Example 2 *The function*

$$d: \quad \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$
$$(a, b) \to |a - b|$$

is a metric on the set $\mathbb{R}$ since
(i) $d(a, b) = |a - b| \geq 0$ for all $a, b \in \mathbb{R}$, and $d(a, b) = 0 \Leftrightarrow |a - b| = 0 \Leftrightarrow a - b = 0 \Leftrightarrow a = b$.
(ii) $d(a, b) = |a - b| = |b - a| = d(b, a)$, and
(iii) $d(a, c) = |a - c| = |a - b + b - c| \leq |a - b| + |b - c| = d(a, b) + d(b, c)$.
(this is deduced from the inequality $|x + y| \leq |x| + |y|$).
*The distance d considered here is known as the **Euclidean metric** on $\mathbb{R}$.*

Example 3 *Similar to the previous example, we can show that the function*

$$\begin{aligned} d :\ & \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \\ & ((a_1, a_2), (b_1, b_2)) \to \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2} \end{aligned}$$

is a metric on $\mathbb{R}^2$*. It is called the* ***Euclidean metric on*** $\mathbb{R}^2$*.*

Example 4 *Let* X *be a non-empty set and* d *the function from* $X \times X$ *into* $\mathbb{R}$ *defined by*

$$d(a, b) = \begin{cases} 0 \text{ if } a = b \\ 1 \text{ if } a \neq b. \end{cases}$$

Then, d *is a metric on* X *and is called the* ***discret metic.***

Example 5 *We can define another metric on* $\mathbb{R}^2$ *by choosing*

$$d^*((a_1, a_2), (b_1, b_2)) = \max\{|a_1 - b_1|, |a_2 - b_2|\}.$$

This form of distance can be generalised to n*-dimensions, i.e.* $\mathbb{R}^n$ *as*

$$d^*(A, B) = \max_{i=\overline{1,n}} \{|a_i - b_i|\},$$

where $A = (a_1, a_2, ..., a_n)$ *and* $B = (b_1, b_2, ..., b_n)$*.*

Example 6 *Yet another metric on* $\mathbb{R}^2$ *is given by*

$$d_1((a_1, a_2), (b_1, b_2)) = |a_1 - b_1| + |a_2 - b_2|,$$

which can also be generalized to $\mathbb{R}^n$ *as*

$$d^*(A, B) = \sum_{i=1}^{n} |a_i - b_i|,$$

where $A = (a_1, a_2, ..., a_n)$ *and* $B = (b_1, b_2, ..., b_n)$*.*

Example 7 *We can also define what is known as the Holder metric on* $\mathbb{R}^n$ *by*

$$d_p(A, B) = \sqrt[p]{\sum_{i=1}^{n} |a_i - b_i|^p},$$

with $p \in [p, \infty)$*.*

Many important examples of metric spaces are "function spaces". For these, the set X on which we put a metric is a set of functions. The following are some examples concerning function spaces.

Example 8 *Let $C[0,1]$ denote the set of continuous functions from $[0,1]$ into $\mathbb{R}$. The following metric may be defined on this set*

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx,$$

where f and g are in $C[0,1]$.

Example 9 *For the same set $C[0,1]$ defined in the previous example, we define another metric as follows*

$$d^*(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Definition 10 *Let (X,d) be a metric space and r any positive real number. The* ***open ball*** *about $a \in X$ of radius r is the set*

$$B_r(a) = \{x \in X |\ d(x,a) < r\}.$$

Example 11 *In the metrix space form by $\mathbb{R}$ and the Euclidean metric, $B_r(a)$ is the open interval $(a-r, a+r)$.*

Example 12 *In $\mathbb{R}^2$ with the Euclidean metric, $B_r(a)$ becomes the open disc with center a and radius r.*

Example 13 *In $\mathbb{R}^2$ with the metric d^*given by*

$$d^*((a_1,a_2),(b_1,b_2)) = \max\{|a_1 - b_1|, |a_2 - b_2|\},$$

the open ball $B_1((0,0))$ is the square plate depicted in Figure???.

Example 14 *In $\mathbb{R}^2$ with the metric d_1 given by*

$$d_1((a_1,a_2),(b_1,b_2)) = |a_1 - b_1| + |a_2 - b_2|,$$

the open ball $B_1((0,0))$ is the diamond shape depicted in Figure???.

Corollary 15 *Let (X, d) be a metric space and B_1 and B_2 open balls in (X, d). Then, $B_1 \cap B_2$ is a union of open balls in (X, d).*

Proposition 16 *Let (X, d) be a metric space. The collection of open balls in (X, d) is a basis for a topology τ on X.*

Example 17 *If d is the Euclidean metric on $\mathbb{R}$, then a basis for the topology τ induced by the metric d is the set of all open balls defined by*

$$B_\delta(a) = (a - \delta, a + \delta).$$

Definition 18 *Metrics on a set X are said to be equivalent if they induce the same topology on X.*

Example 19 *The metrics d, d^*, d_1 defined on $\mathbb{R}^2$ in examples 3, 5, and 6, respectively, are equivalent.*

Proposition 20 *Let (X, d) be a metric space and τ the topology induced on X by the metric d. A sub set U of X is open in (X, τ) if and only if $\forall a \in U$, $\exists \epsilon > 0$ such that the open ball $B_\epsilon(a) \subset U$.*

Proposition 21 *If (X, d) is a metric space and τ is the topology induced on X by d, then (X, τ) is a Hausdorff space (T_2–space) defined as*

$$\forall a, b \in X; a \neq b,\ \exists U, V \in \tau \text{ such that } a \in U,\ b \in V, \text{ and } U \cap V = \emptyset. \tag{1.1}$$

Proof. Since $a, b \in X; a \neq b$, then $d(a, b) = \epsilon > 0$. We can define the two open balls $U = B_{\frac{\epsilon}{2}}(a)$ and $V = B_{\frac{\epsilon}{2}}(b)$, which satisfy (1.1). ■

1.1 Convergence of Sequences

Definition 22 *Let (X, d) be a metric space and $(x_n)_{n \in \mathbb{N}}$ a sequence of points in X. The sequence is said to converge to $x \in X$ if*

$$\lim_{n \to \infty} d(x_n, x) = 0 \Leftrightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow d(x_n, x) < \epsilon.$$

Example 23 *consider the metric space* $(\mathbb{R}, |\cdot|)$. *The sequence* $x_n = 1 + \frac{1}{n}$ *converges to* $x = 1$. *First, we have*

$$\begin{aligned} d(x_n, x) &= \left|1 + \frac{1}{n} - 1\right| \\ &= \frac{1}{n}. \end{aligned}$$

Then, the limist of the distance as n *approaches infinity is given by*

$$\begin{aligned} \lim_{n\to\infty} d(x_n, x) &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0. \end{aligned}$$

Therefore, (x_n) *is convergent towards* 1 *in* $(\mathbb{R}, |\cdot|)$.

Proposition 24 *Any sequence* $(x_n)_{n\in\mathbb{N}}$ *defined in a metric space* (X, d) *is at most convergent towards a unique point, i.e.*

$$\begin{cases} \lim_{n\to\infty} d(x_n, x) = 0 \\ \lim_{n\to\infty} d(x_n, y) = 0 \end{cases} \Rightarrow x = y. \tag{1.2}$$

Proof. The aim is to prove (1.2). Assuming that $\lim_{n\to\infty} d(x_n, x) = 0$ leads to

$$\forall\epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow d(x_n, x) < \epsilon, \tag{1.3}$$

and similarly $\lim_{n\to\infty} d(x_n, y) = 0$ yields

$$\forall\epsilon > 0, \exists n_1 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_1 \Rightarrow d(x_n, y) < \epsilon. \tag{1.4}$$

Hence, for any $n \geq \max\{n_0, n_1\}$, (1.3) and (1.4) imply that $d(x_n, x) < \epsilon$ and $d(x_n, y) < \epsilon$, which can be rewritten as

$$d(x_n, x) + d(x_n, y) < 2\epsilon.$$

Futher simplification yields

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < 2\epsilon.$$

Therefore,

$$\forall\epsilon > 0 : d(x, y) < 2\epsilon \Rightarrow d(x, y) = 0 \Rightarrow x = y.$$

■

Proposition 25 *Let (X, d) be a metric space. A subset A of X is said to be closed in (X, d) iff every convergent sequence of points in A converges to a point in A.*

Example 26 *The subset $A = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ of $\mathbb{R}$ is not closed in $(\mathbb{R}, |\cdot|)$ since $\forall n \in \mathbb{N} : \frac{1}{n} \in A, \ \lim\limits_{n\to\infty} \frac{1}{n} = 0 \notin A$.*

Example 27 *The subset $\mathbb{Q}$ of $\mathbb{R}$ is not closed in $(\mathbb{R}, |\cdot|)$ as the sequence $\left(1 + \frac{1}{n}\right)^n \in \mathbb{Q}$ converges towards $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \notin \mathbb{Q}$.*

Proposition 28 *Let (X, d) and (Y, d') be metric spaces and f a mapping of X into Y, then f is continuous at $x_0 \in X$ iff $\lim\limits_{d(x,x_0)\to 0} d'(f(x), f(x_0)) = 0$, i.e.*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x : d(x, x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \epsilon.$$

Example 29 *Consider the mapping*

$$f \ \ (\mathbb{R}, |\cdot|) \to (\mathbb{R}, |\cdot|)$$
$$x \to f(x),$$

which is continuous at x_0 iff $\lim\limits_{|x-x_0|\to 0} |f(x) - f(x_0)| = 0$, which is equivalent to $\lim\limits_{x\to x_0} f(x) = f(x_0)$, i.e.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Exercise 30 *Let $X = \mathbb{R}$ and $d(x, y) = \begin{cases} 1; x \neq y \\ 0; x = y \end{cases}$. Is the sequence $\left(\frac{1}{n}\right)$ convergent? Justify your answer.*

1.2 Cauchy Sequences

Definition 31 (Cauchy sequence) *Let (X, d) be a metric space. The sequence $\{x_n\} \subset X$ is called a Cauchy sequence if*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

which can be written as

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n, m \in \mathbb{N} : n, m \geq n_0 \Rightarrow d(x_n, x_m) < \epsilon.$$

Proposition 32 *In $(\mathbb{R}, |\cdot|)$, the sequence is Cauchy sequence iff it is a convergent.*

Example 33 *Consider the metric space (X, d), where $X = \mathbb{N}$ and $d(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right|$. We want to show that the sequence $\{n\}$ is a Cauchy sequence but that it is not convergent. We have*

$$\begin{aligned} d(x_n, x_m) &= d(n, m) \\ &= \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m}. \end{aligned}$$

Taking the limist as n and m approch infinity yields

$$0 \leq \lim_{n,m\to\infty} d(x_n, x_m) \leq \lim_{n,m\to\infty} \left(\frac{1}{n} + \frac{1}{m}\right) = 0.$$

Therefore,

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

which proves that $\{n\}$ is a Cauchy sequence. However, assume that the sequence converge towards a value $a \in \mathbb{N}$, i.e.

$$\begin{aligned} \lim_{n\to\infty} d(x_n, a) &= \lim_{n\to\infty} d(n, a) \\ &= \frac{1}{a} \neq 0, \end{aligned}$$

which is a contradiction. Hence, the sequence is divergent.

1.3 Complete Metric Spaces

Definition 34 *A metric space is called complete if every Cauchy sequence defined in it converges to an element of the space.*

Example 35 *The metric space (X, d) with $X = \mathbb{N}$ and $d(n, m) = \left|\frac{1}{n} - \frac{1}{m}\right|$ is not complete.*

Example 36 *The metric space $((0, 1], |\cdot|)$ is not complete since, for instance, the sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence that converges to the point $0 \notin (0, 1]$.*

Example 37 *The metric space $(\mathbb{R}, |\cdot|)$ is complete following Proposition 32.*

1.4 Some Applications

Are the following applications valid distances (metric)?

- $X = \mathbb{R},\ d(x,y) = |x^2 - y^2|$. (not metric)
- $X = \left[0, \frac{\pi}{2}\right],\ d(x,y) = \sin|x-y|$. (metric)
- $X = \left[0, \frac{\pi}{2}\right],\ d(x,y) = \cos|x-y|$. (not metric)
- $X = \mathbb{N},\ d(n,m) = |n-m|$. (metric)
- $X = \mathbb{N},\ d(n,m) = \left|\frac{1}{n} - \frac{1}{m}\right|$. (metric)
- $X = \mathbb{N},\ d(n,m) = \begin{cases} 1; n \neq m \\ 0; n = m \end{cases}$. (metric)
- $X = C[a,b]$ is the space of continuous function on $[a,b]$ and $d(f,g) = \begin{cases} 1; f \neq g \\ 0; f = g \end{cases}$. (metric)
- $X = C_0$ is the space of all sequences that converge to 0 and $d(x,y) = \sup\limits_{n\in\mathbb{N}} |x_n - y_n|$ such that $x = (x_n)$ and $y = (y_n)$. (metric)
- $X = \ell_p : p \in [1,\infty)$ is the space of all sequences (x_n) such that $\sum\limits_{n=1}^{\infty} |x_n|^p < \infty$ (series is convergent) and $d_p(x,y) = \left(\sum\limits_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}}$. (metric)
- $X = \ell_\infty$ is the space of all bounded sequences and $d_\infty(x,y) = \sup\limits_{n\in\mathbb{N}} |x_n - y_n|$ such that $x = (x_n)$ and $y = (y_n)$. (metrix)
  Also prove that
  $$\ell_1 \subset \ell_2 \subset .... \subset C_0 \subset \ell_\infty$$
  and
  $$d_\infty(x,y) = \lim_{n\to\infty} d_p(x,y).$$

- $X = L_p[a,b]$ is the space of p-integrable functions on $[a,b]$, i.e. $\int_a^b |f(x)|^p \, dx < \infty$, and $d_p(f,g) = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}$. (metric)
  If $f(x) = \frac{1}{x}, g(x) = \frac{2}{x} \in L_1[1,4]$, find the distance $d_1\left(\frac{1}{x}, \frac{2}{x}\right)$.

- $X = L_\infty[a,b]$ is the space of essentially bounded functions on $[a,b]$ and $d_\infty(f,g) = \underset{x\in[a,b]}{\operatorname{esssup}} |f(x) - g(x)|$. (metric)
  Also prove that

$$L_\infty[a,b] \subset .... \subset L_2[a,b] \subset L_1[a,b].$$

- Let (X,d) be a metric space. Prove that $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is also a distance.

Chapter 2

Normed Space

Definition 38 *Let X be a linear space. The function*

$$\begin{aligned} \|\cdot\| : \quad & X \to \mathbb{R} \\ & x \to \|x\| \end{aligned}$$

is said to be a ***norm*** *if it satisfies the conditions*
(1) $\forall x \in X : \|x\| \geq 0$ *and* $\|x\| = 0 \Leftrightarrow x = 0.$
(2) $\forall x \in X, \forall \lambda \in \mathbb{R}$ *or* $\mathbb{C} : \|\lambda x\| = |\lambda| \|x\| .$
(3) $\forall x, y \in X : \|x + y\| \leq \|x\| + \|y\| .$
In this case $(X, \|\cdot\|)$ *is called a* ***normed space****.*

Remark 39 *The second condition implies*

$$\|-x\| = \|x\| .$$

Example 40 *Let* $X = \mathbb{R}$ *and* $\|x\| = |x|$*, then* $(\mathbb{R}, |\cdot|)$ *is a normed space since*
(1) $\forall x \in \mathbb{R} : |x| \geq 0$ *and* $|x| = 0 \Leftrightarrow x = 0.$
(2) $\forall x \in \mathbb{R}, \forall \lambda \in \mathbb{R} : |\lambda x| = |\lambda| |x| .$
(3) $\forall x, y \in \mathbb{R} : |x + y| \leq |x| + |y| .$

Example 41 *Let* $X = C[a, b]$ *and* $\|f\| = \sup\limits_{t \in [a,b]} |f(t)|$*, then* $(X, \|\cdot\|)$ *is a normed space as shown below:*
(1) $\forall f \in X : \|f\| = \sup\limits_{t \in [a,b]} |f(t)| \geq 0$ *and*

$$\begin{aligned}\|f\| &= 0 \Leftrightarrow \sup_{t\in[a,b]} |f(t)| = 0 \\ &\Leftrightarrow \forall t \in [a,b] : |f(t)| = 0 \\ &\Leftrightarrow \forall t \in [a,b] : f(t) = 0 \\ &\Leftrightarrow f \equiv 0.\end{aligned}$$

(2) $\forall f \in X,\ \forall \lambda \in \mathbb{R}$ *we have*

$$\begin{aligned}\|\lambda f\| &= \sup_{t\in[a,b]} |\lambda f(t)| \\ &= \sup_{t\in[a,b]} |\lambda| |f(t)| \\ &= |\lambda| \sup_{t\in[a,b]} |f(t)| \\ &= |\lambda| \|f\|.\end{aligned}$$

(3) $\forall f, g \in X$ *we have*

$$\begin{aligned}\|f+g\| &= \sup_{t\in[a,b]} |(f+g)(t)| \\ &= \sup_{t\in[a,b]} |f(t)+g(t)| \\ &\le \sup_{t\in[a,b]} |f(t)| + \sup_{t\in[a,b]} |g(t)| \\ &= \|f\| + \|g\|.\end{aligned}$$

For instance, consider the function

$$f(t) = t - \cos t \in C[0,\pi].$$

The defined norm is given by

$$\|f\| = \sup_{t\in[0,\pi]} |t - \cos t|.$$

In order to calculate the supremum, first, take the derivative $f'(t) = 1 - \sin x > 0$*, meaning that* f *is increasing. Hence,*

$$\begin{aligned}\|f\| &= \sup_{t\in[0,\pi]} |t - \cos t| \\ &= \pi - \cos\pi \\ &= \pi + 1.\end{aligned}$$

Proposition 42 *Let* $(X, \|\cdot\|)$ *be a normed space, then*

$$\forall x, y \in X : |\|x\| - \|y\|| \le \|x - y\|.$$

Proof. First, we know that

$$\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|,$$

which can be rewritten as

$$\|x\| - \|y\| \le \|x - y\|. \tag{2.1}$$

Similarly, we have

$$\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|,$$

yielding

$$-\|x - y\| \le \|x\| - \|y\| \tag{2.2}$$

From (2.1) and (2.2) we get

$$-\|x - y\| \le \|x\| - \|y\| \le \|x - y\|,$$

which simplifies to

$$|\|x\| - \|y\|| \le \|x - y\|.$$

■

Example 43 *Consider the metric space formed by* $X = L_p[a,b]$ *and* $\|x\| = \left(\int_a^b |x(t)|^p \, dx\right)^{\frac{1}{p}}$*, then* $(L_p[a,b], \|\cdot\|)$ *is a normed space as proven below:*

(1) $\forall x \in X : \|x\| = \left(\int_a^b |x(t)|^p \, dx\right)^{\frac{1}{p}} \geq 0,$ *and*

$$\begin{aligned}
\|x\| &= 0 \Leftrightarrow \left(\int_a^b |x(t)|^p \, dx\right)^{\frac{1}{p}} = 0 \\
&\Leftrightarrow \int_a^b |x(t)|^p \, dx = 0 \\
&\Leftrightarrow \forall t \in [a,b] : |x(t)|^p = 0 \\
&\Leftrightarrow \forall t \in [a,b] : |x(t)| = 0 \\
&\Leftrightarrow \forall t \in [a,b] : x(t) = 0 \\
&\Leftrightarrow x \equiv 0.
\end{aligned}$$

(2) $\forall x \in X, \ \forall \lambda \in \mathbb{R},$ *then*

$$\begin{aligned}
\|\lambda x\| &= \left(\int_a^b |\lambda x(t)|^p \, dx\right)^{\frac{1}{p}} \\
&= \left(|\lambda|^p \int_a^b |x(t)|^p \, dx\right)^{\frac{1}{p}} \\
&= |\lambda| \left(\int_a^b |x(t)|^p \, dx\right)^{\frac{1}{p}} \\
&= |\lambda| \, \|x\| .
\end{aligned}$$

.

(3) $\forall x, y \in X,$ *we have*

$$\begin{aligned}
\|x + y\| &= \left(\int_a^b |(x+y)(t)|^p \, dx\right)^{\frac{1}{p}} \\
&= \left(\int_a^b |x(t) + y(t)|^p \, dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_a^b |x(t)|^p \, dx\right)^{\frac{1}{p}} + \left(\int_a^b |y(t)|^p \, dx\right)^{\frac{1}{p}} \\
&= \|x\| + \|y\| .
\end{aligned}$$

Take, for instance, the function $x(t) = \frac{1}{x} \in L_2[1,4]$, *then*

$$\begin{aligned} \|x\| &= \left(\int_1^4 |x(t)|^2 \, dx \right)^{\frac{1}{2}} \\ &= \left(\int_1^4 \left| \frac{1}{x} \right|^2 dx \right)^{\frac{1}{2}} \\ &= \frac{1}{2}\sqrt{3}. \end{aligned}$$

Example 44 *Let* $X = C_0$ *be the space of all sequences that converge to* 0 *and* $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$, $x = (x_n)$ $(C_0, \|\cdot\|)$, *then* $(C_0, \|\cdot\|)$ *is a normed space.*
For instance, consider the sequences

$$x = \left(\frac{1}{n}\right) = \left\{1, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}, ...\right\} \in C_0,$$

with the norm

$$\begin{aligned} \|x\| &= \sup_{n\in\mathbb{N}} \left| \frac{1}{n} \right| \\ &= 1, \end{aligned}$$

and

$$y = \left(\frac{1}{2^n}\right) = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ..., \frac{1}{n}, ...\right\} \in C_0,$$

with the norm

$$\begin{aligned} \|y\| &= \sup_{n\in\mathbb{N}} \left| \frac{1}{2^n} \right| \\ &= \frac{1}{2}. \end{aligned}$$

Example 45 *The metric space* $(X, \|\cdot\|)$ *in each of the following three cases is a normed space:*

$$\begin{aligned} &X = \ell_\infty, && \|x\| = \sup_{n\in\mathbb{N}} |x_n| \\ &X = \ell_p, && \|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}} \\ &X = L_p[a,b] && \|x\| = \operatorname*{esssup}_{t\in[a,b]} |x(t)|. \end{aligned}$$

Note that there is a difference between the supremum (sup) *and essential supremum* (esssup)*. For instance, if*

$$x(t) = \begin{cases} 1, t \in [0,1) \\ 2, t = 1, \end{cases}$$

then

$$\operatorname*{esssup}_{t\in[0,1]} |x(t)| = 1,$$

but

$$\sup_{t\in[0,1]} |x(t)| = 2.$$

Example 46 *Consider the sequence*

$$x = \frac{3}{\sqrt{n(n+1)}} \in \ell_2.$$

The following is a valid norm for ℓ_2

$$\begin{aligned} \|x\| &= \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{n=1}^{\infty} \frac{9}{n(n+1)}\right)^{\frac{1}{2}} \\ &= 3\left(\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\right)^{\frac{1}{2}} \\ &= 3\left(\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)\right)^{\frac{1}{2}} \\ &= 3. \end{aligned}$$

Remark 47 *Note that in the definition of a normed space, it was expilicitly required that the space be a linear (vector) space. However, this is not a condition for the space to be metric. Thus, a metric space is not necessarily a normed space.*

Theorem 48 *Every normed space is a metric space.*

Proof. Let $(X, \|\cdot\|)$ be a normed space and define $d(x, y) = \|x - y\|$. We have:

(1) $d(x, y) = \|x - y\| \geq 0$ and $d(x, y) = 0 \Leftrightarrow \|x - y\| = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$.

(2) $d(x, y) = \|x - y\| = \|y - x\| = d(y, x)$.

(3) $d(x, y) = \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| = d(x, z) + d(z, y)$.

It simply follows that (X, d) is a metric space. ■

2.1 Convergent Sequences

Definition 49 *A sequence $\{x_n\}$ is convergent to a point a in the normed space $(X, \|\cdot\|)$ iff:*

$$\lim_{n\to\infty} \|x_n - a\| = 0,$$

which can be written as

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow \|x_n - a\| < \epsilon.$$

2.2 Banach Spaces

Definition 50 *The normed space $(X, \|\cdot\|)$ is called a Banach space if every Cauchy sequence is convergence in X.*

Chapter 3

Inner Product Spaces

3.1 The Real Case

Definition 51 *Let X be a vector space on $\mathbb{R}$. The inner product $\langle x, y\rangle$ such that*

$$\langle \cdot, \cdot\rangle \;\; X \times X \to \mathbb{R}$$

$$(x, y) \to \langle x, y\rangle\,,$$

satisfies the following conditions:
(1) $\forall x \in X : \langle x, x\rangle \geq 0$ *and* $\langle x, x\rangle = 0 \Leftrightarrow x = 0.$
(2) $\forall x, y \in X : \langle x, y\rangle = \langle y, x\rangle\,.$
(3) $\forall x, y \in X,\ \forall \lambda \in \mathbb{R} : \langle \lambda x, y\rangle = \lambda \langle x, y\rangle\,.$
(4) $\forall x, y, z \in X : \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle\,.$
The space $(X, \langle \cdot, \cdot\rangle)$ is called an inner product space.

Remark 52 *Based on the properties stated in Definition 51, we have:*
(1) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle\,.$
(2) $\langle x, \alpha y\rangle = \alpha \langle x, y\rangle\,.$
(3) $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta \langle x, z\rangle\,.$

Example 53 *Let $X = \mathbb{R}$ and $\langle x, y\rangle = xy$. Then, $(\mathbb{R}, \langle \cdot, \cdot\rangle)$ is an inner product space as it satisfies the four conditions:*
(1) $\forall x \in \mathbb{R} : \langle x, x\rangle = x^2 \geq 0$ *and*

$$\langle x, x\rangle = 0 \Leftrightarrow x^2 = 0 \Leftrightarrow x = 0.$$

(2) $\forall x, y \in \mathbb{R} : \langle x, y \rangle = xy = yx = \langle y, x \rangle$.
(3) $\forall x, y, \lambda \in \mathbb{R} : \langle \lambda x, y \rangle = \lambda xy = \lambda \langle x, y \rangle$.
(4) $\forall x, y, z \in \mathbb{R} : \langle x + y, z \rangle = (x + y) z = xz + yz = \langle x, z \rangle + \langle y, z \rangle$.

Example 54 *Let* $X = L_2 [a, b]$ *and* $\langle f, g \rangle = \int_a^b f(t) g(t) \, dt$. *We have:*
(1) $\langle f, f \rangle = \int_a^b (f(t))^2 \, dt \geq 0$ *and*

$$\begin{aligned} \langle f, f \rangle &= 0 \Leftrightarrow \int_a^b (f(t))^2 \, dt = 0 \\ &\Leftrightarrow \forall t \in [a, b] : (f(t))^2 = 0 \\ &\Leftrightarrow \forall t \in [a, b] : f(t) = 0 \\ &\Leftrightarrow f \equiv 0. \end{aligned}$$

(2) $\langle f, g \rangle = \int_a^b f(t) g(t) \, dt = \int_a^b g(t) f(t) \, dt = \langle g, f \rangle$.
(3) $\langle \lambda f, g \rangle = \int_a^b \lambda f(t) g(t) \, dt = \lambda \int_a^b f(t) g(t) \, dt = \lambda \langle f, g \rangle$.
(4) $\langle f + g, h \rangle = \int_a^b (f + g)(t) h(t) \, dt = \int_a^b [f(t) + g(t)] h(t) \, dt = \int_a^b f(t) h(t) \, dt + \int_a^b g(t) h(t) \, dt = \langle f, h \rangle + \langle g, h \rangle$.
Therefore, $(L_p [a, b], \langle \cdot, \cdot \rangle)$ *is an inner product space.*

Example 55 *Let* $X = \ell_2$ *and* $\langle x, y \rangle = \sum_{n=1}^{\infty} |x_n y_n|$. *Is* $(X, \langle \cdot, \cdot \rangle)$ *an inner product space? We have:*

(1) $\forall x \in \ell_2 : \langle x, x \rangle = \sum_{n=1}^{\infty} x_n^2 \geq 0$ and

$$\begin{aligned} \langle x, x \rangle &= 0 \Leftrightarrow \sum_{n=1}^{\infty} x_n^2 = 0 \\ &\Leftrightarrow \forall n \in \mathbb{N} : x_n^2 = 0 \\ &\Leftrightarrow \forall n \in \mathbb{N} : x_n = 0 \\ &\Leftrightarrow x = 0. \end{aligned}$$

(2) $\forall x, y \in \ell_2 : \langle x, y \rangle = \sum_{n=1}^{\infty} |x_n y_n| = \sum_{n=1}^{\infty} |y_n x_n| = \langle y, x \rangle$.

(3) $\forall x, y \in \ell_2, \forall \lambda \in \mathbb{R} : \langle \lambda x, y \rangle = \sum_{n=1}^{\infty} |\lambda x_n y_n| = |\lambda| \sum_{n=1}^{\infty} |x_n y_n| = |\lambda| \langle x, y \rangle \neq \lambda \langle x, y \rangle$ if $\lambda < 0$.
Hence, $\langle \cdot, \cdot \rangle$ is not an inner product and, consequently, $(\ell_2, \langle \cdot, \cdot \rangle)$ is not an inner product space.

Example 56 *Let $X = \ell_2$ and $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n$. Is $(X, \langle \cdot, \cdot \rangle)$ an inner product space?*
(1) $\forall x \in \ell_2 : \langle x, x \rangle = \sum_{n=1}^{\infty} x_n^2 \geq 0$ and

$$\begin{aligned}
\langle x, x \rangle &= 0 \Leftrightarrow \sum_{n=1}^{\infty} x_n^2 = 0 \\
&\Leftrightarrow \forall n \in \mathbb{N} : x_n^2 = 0 \\
&\Leftrightarrow \forall n \in \mathbb{N} : x_n = 0 \\
&\Leftrightarrow x = 0.
\end{aligned}$$

(2) $\forall x, y \in \ell_2 : \langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} y_n x_n = \langle y, x \rangle$.
(3) $\forall x, y \in \ell_2, \forall \lambda \in \mathbb{R} : \langle \lambda x, y \rangle = \sum_{n=1}^{\infty} \lambda x_n y_n = \lambda \sum_{n=1}^{\infty} x_n y_n = \lambda \langle x, y \rangle$.
(4) $\forall x, y, z \in \ell_2 : \langle x + y, z \rangle = \sum_{n=1}^{\infty} (x_n + y_n) z_n = \sum_{n=1}^{\infty} x_n z_n + \sum_{n=1}^{\infty} y_n z_n = \langle x, z \rangle + \langle y, z \rangle$.
Therefore, $(\ell_2, \langle \cdot, \cdot \rangle)$ is an inner product space.

Example 57 *Consider the two functions $f(t) = t$ and $g(t) = \cos t$ in $L_2[0, \pi]$. The inner product as defined in Example 54 can be calculated as*

$$\begin{aligned}
\langle f, g \rangle &= \int_0^{\pi} f(t)\, g(t)\, dt \\
&= \int_0^{\pi} t \cos t dt \\
&= t \sin t]_0^{\pi} - \int_0^{\pi} \sin t dt \\
&= \cos t]_0^{\pi} = -2.
\end{aligned}$$

Example 58 *Consider the two sequences*

$$x = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\} \in \ell_2,$$

and

$$y = \{1, 1, 0, ...\} \in \ell_2.$$

The inner product of the two sequences as defined in Example 56 is given by

$$\begin{aligned} \langle x, y \rangle &= \sum_{n=1}^{\infty} x_n y_n \\ &= 1 + \frac{1}{2} + 0 + 0... \\ &= \frac{3}{2}. \end{aligned}$$

Proposition 59 *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space and define*

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Then,

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.$$

Proof. Let

$$\lambda = \frac{\langle x, y \rangle}{\langle y, y \rangle} = \frac{\langle x, y \rangle}{\|y\|^2}.$$

We have $0 \leq \|x - \lambda y\|^2 = \langle x - \lambda y, x - \lambda y \rangle$

$$\begin{aligned} \|x - \lambda y\|^2 &= \langle x - \lambda y, x - \lambda y \rangle \\ &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\ &= \|x\|^2 - 2\lambda \langle x, y \rangle + \lambda^2 \|y\|^2 \\ &= \|x\|^2 - 2\left(\frac{\langle x, y \rangle}{\|y\|^2}\right) \langle x, y \rangle + \left(\frac{\langle x, y \rangle}{\|y\|^2}\right)^2 \|y\|^2 \\ &= \|x\|^2 - 2\frac{\langle x, y \rangle^2}{\|y\|^2} + \frac{\langle x, y \rangle^2}{\|y\|^2} \\ &= \|x\|^2 - \frac{\langle x, y \rangle^2}{\|y\|^2} \geq 0, \end{aligned}$$

which is rearranged to produce

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\| .$$

■

Proposition 60 *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. Then, $(X, \|\cdot\|)$ is a normed space where $\|x\| = \sqrt{\langle x, x\rangle}$.*

Proof. We aim to prove that $\|x\| = \sqrt{\langle x, x\rangle}$is a valid norm by checking the conditions in Definition **??**:

(1) We have

$$\forall x \in X : \langle x, x\rangle \geq 0 \Rightarrow \sqrt{\langle x, x\rangle} \geq 0 \Rightarrow \|x\| \geq 0,$$

and

$$\|x\| = 0 \Leftrightarrow \sqrt{\langle x, x\rangle} = 0 \Leftrightarrow \langle x, x\rangle = 0 \Leftrightarrow x = 0.$$

(2) We also have

$$\begin{aligned} \|\lambda x\| &= \sqrt{\langle \lambda x, \lambda x\rangle} \\ &= \sqrt{\lambda^2 \langle x, x\rangle} \\ &= |\lambda| \sqrt{\langle x, x\rangle} \\ &= |\lambda| \, \|x\| . \end{aligned}$$

(3) Lastly,

$$\begin{aligned} \|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\| \cdot \|y\| \\ &= (\|x\| + \|y\|)^2 , \end{aligned}$$

which yields

$$\|x+y\| \leq \|x\| + \|y\| .$$

Therefore, $\|\cdot\|$ is a norm and $(X, \|\cdot\|)$ is a normed space. ■

Proposition 61 *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space and define $\|x\| = \sqrt{\langle x, x\rangle}$. Then,*

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) . \tag{3.1}$$

Remark 62 *Not all norms are induced by an inner product. For a valid norm to be induced by an inner product, it has to satisfy the equality 3.1.*

Proof. Counter example: Consider the normed space $(C[0,1], \|\cdot\|)$ where $\|x\| = \sup_{t\in[0,1]} |x(t)|$ and defined the functions

$$x(t) = 1,\ y(t) = t;\ t \in [0,1],$$

which are clearly elements of $C[0,1]$. We have

$$\|x\| = 1,\ \|y\| = \sup_{t\in[0,1]} |t| = 1,$$

leading to

$$\begin{aligned} \|x+y\| &= \sup_{t\in[0,1]} |x(t) + y(t)| \\ &= \sup_{t\in[0,1]} |1+t| \\ &= 2, \end{aligned}$$

and

$$\begin{aligned} \|x-y\| &= \sup_{t\in[0,1]} |x(t) - y(t)| \\ &= \sup_{t\in[0,1]} |1-t| \\ &= 1. \end{aligned}$$

Since

$$\|x+y\|^2 + \|x-y\|^2 = 5 \neq 4 = 2\left(\|x\|^2 + \|y\|^2\right),$$

then, $C[0,1]$ is not an inner product space.

Counter example: Consider the normed space $(\ell_\infty, \|\cdot\|)$ where $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$ and let

$$x = (0,1,0,1,....) \in \ell_\infty,\ \|x\| = 1,$$

and

$$y = (1,0,1,0,....) \in \ell_\infty,\ \|y\| = 1.$$

We have

$$x + y = (1,1,1,1,....) \in \ell_\infty,\ \|x+y\| = 1,$$

and

$$x - y = (-1, 1, -1, 1, ...) \in \ell_\infty, \ \|x - y\| = 1,$$

leading to

$$\|x + y\|^2 + \|x - y\|^2 = 2 \neq 4 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Counter example: Consider the normed space $(\ell_p, \|\cdot\|)$ where $\|x\| = \left(\sum\limits_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$; $p \neq 2$, and define

$$x = (1, 1, 0, 0, ...), \ y = (1, -1, 0, 0, ...) \in \ell_p.$$

The sum in difference are given by

$$x + y = (2, 0, 0, 0, ...),$$

and

$$x - y = (0, 2, 0, 0, ...),$$

respectively. We have the norms

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} = 2^{\frac{1}{p}},$$

$$\|y\| = \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}} = 2^{\frac{1}{p}},$$

$$\|x + y\| = \left(\sum_{n=1}^{\infty} |x_n + y_n|^p\right)^{\frac{1}{p}} = 2,$$

and

$$\|x - y\| = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{\frac{1}{p}} = 2,$$

leading to

$$\|x + y\|^2 + \|x - y\|^2 = 8,$$

and

$$2\left(\|x\|^2 + \|y\|^2\right) = 4 \cdot 2^{\frac{2}{p}}.$$

This leads to a contradiction as

$$4 \cdot 2^{\frac{2}{p}} = 8 \Leftrightarrow p = 2,$$

which in our case yields

$$\|x+y\|^2 + \|x-y\|^2 \neq 2\left(\|x\|^2 + \|y\|^2\right).$$

Thus, $(\ell_p, \|\cdot\|)$ is not an inner product space for $p \neq 2$. ■

3.2 The Complex Case

Definition 63 *Let X be a vector space on $\mathbb{C}$. The inner product $\langle x, y\rangle$ such that*

$$\begin{aligned} \langle \cdot, \cdot \rangle \;\; & X \times X \to \mathbb{C} \\ & (x,y) \to \langle x, y\rangle, \end{aligned}$$

satisfies the following conditions:
(1) $\forall x \in X : \langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.
(2) $\forall x, y \in X : \langle x, y\rangle = \overline{\langle y, x\rangle}$.
(3) $\forall x, y \in X,\ \forall \lambda \in \mathbb{C} : \langle \lambda x, y\rangle = \lambda \langle x, y\rangle$.
(4) $\forall x, y, z \in X : \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
It follows that $(X, \langle \cdot, \cdot\rangle)$ is called an inner product space.

Proposition 64 *The conditions shown in Definition 63 yields the following for $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in X$:*
(1) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$.
(2) $\langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle} = \overline{\alpha \langle y, x\rangle} = \overline{\alpha}\overline{\langle y, x\rangle} = \overline{\alpha}\langle x, y\rangle$.
(3) $\langle x, \alpha y + \beta z\rangle = \overline{\langle \alpha y + \beta z, x\rangle} = \overline{\alpha}\overline{\langle y, x\rangle} + \overline{\beta}\overline{\langle z, x\rangle} = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle$.

Example 65 *Consider the vector space $X = \mathbb{C}$. We define the inner product as*

$$\langle x, y\rangle = x\overline{y}.$$

Then, $(\mathbb{C}, \langle \cdot, \cdot\rangle)$ is an inner product space.
In order to show that $\langle \cdot, \cdot\rangle$ is an inner product, we verify the conditions of Definition 63:
(1) The first condition is $\forall x \in \mathbb{C} : \langle x, x\rangle \geq 0$, which is clear as

$$\begin{aligned} \langle x, x\rangle &= x\overline{x} \\ &= \|x\|_2^2 \geq 0. \end{aligned}$$

Also

$$\langle x, x \rangle = 0 \Leftrightarrow \|x\|_2^2 = 0 \Leftrightarrow x = 0.$$

(2) The second condition is $\forall x, y \in \mathbb{C} : \langle x, y \rangle = \overline{\langle y, x \rangle}$, *which can be verified as*

$$\langle x, y \rangle = x\overline{y} = \overline{y}x = \overline{y\overline{x}} = \overline{\langle y, x \rangle}.$$

(3) The third condition is $\forall \lambda \in \mathbb{C}, \forall x, y \in \mathbb{C}$. *We have*

$$\begin{aligned} \langle \lambda x, y \rangle &= \lambda x \overline{y} \\ &= \lambda \langle x, y \rangle \end{aligned}$$

(4) The last condition is $\forall x, y, z \in \mathbb{C} : \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$, *which is verified by*

$$\langle x + y, z \rangle = (x + y)\overline{z} = x\overline{z} + y\overline{z} = \langle x, z \rangle + \langle y, z \rangle.$$

Example 66 *Let* $X = L_2[a, b]$ *on* $\mathbb{C}$ *and* $f \in X$, *i.e.* $f : [a, b] \to \mathbb{C}$. *We define the inner product as*

$$\langle f, g \rangle = \int_a^b f(t)\,\overline{g(t)}dt.$$

Solution 67 *Similar to the previous example, we verify the four conditions of Definition 63:*
(1) We have

$$\begin{aligned} \langle f, f \rangle &= \int_a^b f(t)\,\overline{f(t)}dt \\ &= \int_a^b \|f(t)\|_2^2\, dt \geq 0, \end{aligned}$$

and

$$\begin{aligned} \langle f, f \rangle &= 0 \Leftrightarrow \int_a^b \|f(t)\|_2^2\, dt = 0 \\ &\Leftrightarrow \|f(t)\|_2^2 = 0 \\ &\Leftrightarrow \|f(t)\|_2 = 0 \\ &\Leftrightarrow f \equiv 0 \ \text{on} \ [a, b]. \end{aligned}$$

(2) We have

$$\begin{aligned}
\langle f, g\rangle &= \int_a^b f(t)\,\overline{g(t)}dt \\
&= \int_a^b \overline{g(t)}f(t)\,dt \\
&= \overline{\int_a^b g(t)\,\overline{f(t)}dt} \\
&= \overline{\langle g, f\rangle}.
\end{aligned}$$

(3) For the third condition:

$$\begin{aligned}
\langle \lambda f, g\rangle &= \int_a^b \lambda f(t)\,\overline{g(t)}dt \\
&= \lambda \int_a^b f(t)\,\overline{g(t)}dt \\
&= \lambda \langle f, g\rangle.
\end{aligned}$$

(4) Lastly,

$$\begin{aligned}
\langle f+g, h\rangle &= \int_a^b [f+g](t)\,\overline{h(t)}dt \\
&= \int_a^b \left[f(t)\,\overline{h(t)} + g(t)\,\overline{h(t)}\right] dt \\
&= \int_a^b f(t)\,\overline{h(t)}dt + \int_a^b g(t)\,\overline{h(t)}dt \\
&= \langle f, h\rangle + \langle g, h\rangle.
\end{aligned}$$

Hence, $(L_2[a,b], \langle\cdot,\cdot\rangle)$ *where* $\langle f, g\rangle = \int_a^b f(t)\,\overline{g(t)}dt$ *is an inner product space.*

Example 68 *Consider the space $X = \ell_2$ with $x = (x_n)_{n\in\mathbb{N}}$ and $y = (y_n)_{n\in\mathbb{N}}$ on $\mathbb{C}$. We define the inner product as*

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}.$$

To verify this, we have:
(1) We have

$$\begin{aligned} \langle x, x\rangle &= \sum_{n=1}^{\infty} x_n \overline{x_n} \\ &= \sum_{n=1}^{\infty} \|x_n\|_2^2 \geq 0, \end{aligned}$$

and

$$\begin{aligned} \langle x, x\rangle &= 0 \Leftrightarrow \sum_{n=1}^{\infty} \|x_n\|_2^2 = 0 \\ &\Leftrightarrow \forall n \in \mathbb{N} : \|x_n\|_2^2 = 0 \\ &\Leftrightarrow \forall n \in \mathbb{N} : \|x_n\|_2 = 0 \\ &\Leftrightarrow \forall n \in \mathbb{N} : x_n = 0 \\ &\Leftrightarrow x \equiv 0. \end{aligned}$$

(2) For the second condition, we have

$$\begin{aligned} \langle x, y\rangle &= \sum_{n=1}^{\infty} x_n \overline{y_n} \\ &= \sum_{n=1}^{\infty} \overline{y_n} x_n \\ &= \overline{\sum_{n=1}^{\infty} y_n \overline{x_n}} \\ &= \overline{\langle y, x\rangle}. \end{aligned}$$

(3) Also,

$$\begin{aligned}\langle \lambda x, y\rangle &= \sum_{n=1}^{\infty} \lambda x_n \overline{y_n} \\ &= \lambda \sum_{n=1}^{\infty} x_n \overline{y_n} \\ &= \lambda \langle x, y\rangle .\end{aligned}$$

(4) Lastly,

$$\begin{aligned}\langle x+y, z\rangle &= \sum_{n=1}^{\infty} [x_n + y_n]\, \overline{z_n} \\ &= \sum_{n=1}^{\infty} [x_n \overline{z_n} + y_n \overline{z_n}] \\ &= \sum_{n=1}^{\infty} x_n \overline{z_n} + \sum_{n=1}^{\infty} y_n \overline{z_n} \\ &= \langle x, z\rangle + \langle y, z\rangle .\end{aligned}$$

Therefore, $(\ell_2, \langle \cdot, \cdot \rangle)$, with $\langle x, y\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$, is an inner product space.

3.3 Hilbert Spaces

Definition 69 *The inner product space $(X, \langle \cdot, \cdot \rangle)$ is called a Hilbert space if it is complete, i.e. every Cauchy sequence is convergent on X.*

Example 70 *The inner product spaces $(L_2[a,b], \langle \cdot, \cdot \rangle)$ and $(\ell_2, \langle \cdot, \cdot \rangle)$ (see Examples 56 and 54) are Hilbert spaces.*

Remark 71 *Since every inner product space is a normed space, every Hilbert space is a Banach space.*

3.4 Orthogonality

Definition 72 *Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space. The two elements $x, y \in X$ are called orthogonal ($x \perp y$) if $\langle x, y\rangle = 0$.*

Example 73 *Consider the space $L_2[0, 2\pi]$, and define the two functions*

$$x(t) = \sin t,$$

and

$$y(t) = \cos t.$$

The inner product of the two functions is given by

$$\begin{aligned}\langle x, y\rangle &= \int_0^{2\pi} x(t)\, y(t)\, dt \\ &= \int_0^{2\pi} \sin t \cos t dt \\ &= \left.\frac{1}{2}\sin^2 x\right]_0^{2\pi} = 0.\end{aligned}$$

Thus, the two functions are orthogonal ($x \perp y$).

Remark 74 *For the inner product space $(X, \langle\cdot,\cdot\rangle)$,*
** The set $\{x_1, x_2, ...\} \subset X$ is called an* ***orthogonal*** *set if*

$$\forall n, m\ \ (n \neq m):\ \langle x_n, x_m\rangle = 0.$$

** The set $\{x_1, x_2, ...\} \subset X$ is called an* ***orthonarmal*** *set if it is orthogonal and*

$$\forall n:\ \langle x_n, x_n\rangle = 1.$$

Example 75 *Consider the set*

$$\{e_1, e_2, ...\} \in \ell_2,$$

where

$$e_1 = (1, 0, 0, 0...),$$

$$e_2 = (0, 1, 0, 0...),$$

$$e_3 = (0, 0, 1, 0...)\,,$$
$$\vdots$$
$$e_n = (0, 0, 0, 0, ...0, 1, 0...)\,.$$

Therefore, since $\langle e_n, e_m\rangle = 0$ *where* $n \neq m$ *then* $\{e_1, e_2, ...\}$ *is orthogonal, and since* $\langle e_n, e_n\rangle = 1$ *then* $\{e_1, e_2, ...\}$ *is orthonormal.*

Lemma 76 *Let* $x, y \in X$ *be orthogonal elements. This is equivalent to*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2\,.$$

Proof. By definition, since x and y are orthogonal, $\langle x, y\rangle = 0$. Therefore,

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y\rangle \\ &= \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \\ &= \|x\|^2 + \|y\|^2\,. \end{aligned}$$

■

3.5 Projections

Definition 77 (Direct Sum) *Let* Y *and* Z *be subspaces of a vector (linear) space* X*. We say that* X *is a direct sum of* Y *and* Z*, denoted by* $X = Y \oplus Z$*, if:*

$$\forall x \in X : \exists y \in Y, z \in Z : x = y + z.$$

Example 78 *Consider the inner product space* $(\mathbb{R}^2, \langle\cdot,\cdot\rangle)$ *with* $\langle x, y\rangle = x_1y_1 + x_2y_2$ *and defined the spaces*

$$X = \mathbb{R}^2 = \{(\alpha, \beta)\,;\ \alpha, \beta \in \mathbb{R}\}\,,$$

$$Y = \mathbb{R} \times \{0\} = \{(\alpha, 0)\,;\ \alpha \in \mathbb{R}\}\,,$$

and

$$Z = \{0\} \times \mathbb{R} = \{(0, \beta)\,;\ \beta \in \mathbb{R}\}\,.$$

We can see that

$$\mathbb{R}^2 = (\mathbb{R} \times \{0\}) \oplus (\{0\} \times \mathbb{R})\,,$$

as

$$\forall x = (\alpha, \beta) \in \mathbb{R}^2 :\ (\alpha, \beta) = (\alpha, 0) + (0, \beta)\,,$$

where $(\alpha, 0) \in (\mathbb{R} \times \{0\})$ *and* $(0, \beta) \in (\{0\} \times \mathbb{R})$*.*

Definition 79 (Orthoplement) *Let* $(X, \langle \cdot, \cdot \rangle)$ *be an inner product space. The orthoplement of* $Y \subset X$*, denoted by* $Y^{\perp}$*, is defined by*

$$Y^{\perp} = \{y \in X : y \perp Y\},$$

where $y \perp Y$ *is equivalent to*

$$\forall x \in Y : y \perp x.$$

Example 80 *Consider the inner product space* $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$*. The orthoplement of* $Y = \mathbb{R} \times \{0\}$ *is given by*

$$Y^{\perp} = \{0\} \times \mathbb{R}.$$

Theorem 81 *Let* Y *be a closed subspace of a Hilbert space* H*, then*

$$H = Y \oplus Y^{\perp},$$

i.e.

$$\forall x \in H, \ \exists y \in Y, z \in Y^{\perp} : y \perp z \text{ and } x = y + z.$$

Definition 82 *The element* $y \in Y$ *is called a projection of* $x \in H$*. In this case we can define a map*

$$\begin{aligned} P : \ & H \to Y \\ & x \to Px = y, \end{aligned}$$

in which case P *is called a projection operator.*

Chapter 4

Operators

Definition 83 *An* ***operator*** A *is a mapping form a set* X *into a set* Y*, that is* $A : X \to Y$.

Example 84 *The function* f *defined by* $f(x) = x^2$ *maps the set* $X = \{1, 2, 5\}$ *into* $Y = \{1, 4, 25\}$*. We write*

$$f : X \to Y$$
$$x \to x^2.$$

Example 85 *The differential operator* $\frac{d}{dt}$ *maps* $X = \{t^2, \sin t, e^{5t}\}$ *into* $Y = \{2t, \cot t, 5e^{5t}\}$.

Example 86 *Among the many types of integral operators, we have:*
(1) The Volttera integral operator $Ax(t) = \int_0^t x(s)\,ds$*, which maps* $X = \{1, t^2, e^{5t}\}$ *into* $Y = \left\{t, \frac{t^3}{3}, \frac{1}{5}\left(e^{5t} - 1\right)\right\}$.
(2) The Fredholm integral $Ax(t) = \int_0^1 x(s)\,ds$*, which maps* $X = \{1, t^2, e^{5t}\}$ *into* $Y = \left\{1, \frac{1}{3}, \frac{1}{5}\left(e^5 - 1\right)\right\}$.
(3) The Laplace integral operator $\triangle(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$ *maps* $X = \{1, t^2, e^{5t}\}$ *into* $Y = \left\{\frac{1}{s}, \frac{1}{s^2}, \frac{1}{s-1}\right\}$.

Definition 87 *An operator* $A : X \to Y$ *where* $X, Y \subset \mathbb{R}$ *is called a function.*

Definition 88 *An operator* $A : X \to \mathbb{R}$ *is called a functional.*

Example 89 *The Fredholm integral operator is an example of a functional.*

Definition 90 *A operator $A : X \to Y$ is said to be linear if X and Y are linear (vector) spaces on $\mathbb{R}$ and*

$$\forall \alpha, \beta \in \mathbb{R},\ \forall x, y \in X:\ A(\alpha x + \beta y) = \alpha A x + \beta A y.$$

Example 91 *The differential operator $\frac{d}{dt}$ is a linear one since*

$$\frac{d}{dt}(\alpha x + \beta y) = \alpha \frac{d}{dt} x + \beta \frac{d}{dt} y.$$

Similarly, the integral operator is linea as

$$\int (\alpha x + \beta y) = \alpha \int x + \beta \int y.$$

Remark 92 *The linear operator $A : X \to \mathbb{R}$ is called a linear functional.*

Example 93 *Consider the functional $A : X \to \mathbb{R}$ such that $Ax = x^2$. We have*

$$\begin{aligned} A(\alpha x + \beta y) &= (\alpha x + \beta y)^2 \\ &= \alpha^2 x^2 + \beta^2 y^2 + 2\alpha\beta x y \\ &\neq \alpha A x + \beta A y \\ &= \alpha x^2 + \beta y^2, \end{aligned}$$

and thus A is not a linear operator.

Example 94 *For the functional $Ax(t) = \int_0^1 |x(s)|^2 \, ds$, we have*

$$\begin{aligned} A(\alpha x + \beta y) &= \int_0^1 (\alpha x + \beta y)^2 \, ds \\ &\neq \alpha \int_0^1 x^2 ds + \beta^2 \int_0^1 y^2 ds. \end{aligned}$$

Hence, A is not a linear functional.

Example 95 *The Laplace operator is linear since*

$$\begin{aligned} \triangle(\alpha x + \beta y) &= \int_0^\infty e^{-st} (\alpha x + \beta y) \, ds \\ &= \int_0^\infty \alpha e^{-st} x ds + \int_0^\infty \beta e^{-st} y ds \\ &= \alpha \int_0^\infty e^{-st} x ds + \beta \int_0^\infty e^{-st} y ds \\ &= \alpha \triangle(x) + \beta \triangle(y). \end{aligned}$$

Definition 96 (Unit operator) *The operator $I : X \to X$ is called a unit operator if $Ix = x$.*

Example 97 *The operator $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a unit operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ as*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Definition 98 (Inverse operator) *The operator $B : Y \to X$ is called a left or right inverse of the operator $A : X \to Y$ if $BA = I$ or $AB = I$, respectively.*
If B is the left and right inverse simultaneously, it is called the inverse, denoted by A^{-1}, i.e.

$$BA = AB = I.$$

Example 99 *Let $A := \frac{d}{dt}$ and $B := \int_0^t$. According to the fundamental theorem of calculus, we have*

$$\frac{d}{dt} \int_0^t f(s)\, ds = f(t),$$

and

$$\int_0^t \frac{d}{ds} f(s)\, ds = f(t) - f(0).$$

Note that A is the left inverse of B but A is not the right inverse of B. Similarly, B is the right inverse of A but B is not the left inverse of A.

Lemma 100 *There exists at most one inverse to any operator.*

Proof. Let $A : X \to Y$. The proof is separated into two parts:

1) If no inverse exists, then we have nothing to prove.

2) If $B, C : Y \to X$ are two inverses of A, then $AC = I$ and $BA = I$, which leads to

$$\begin{aligned} B &= BI \\ &= BAC \\ &= IC \\ &= C. \end{aligned}$$

■

Definition 101 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. The operator*

$$A : X \to Y$$

is said to be:
***Bounded**: If there exists $C > 0$ such that $\forall x \in X$:*

$$\|Ax\|_Y \leq C \|x\|_X .$$

***Continuous**: If $\forall x, y \in X$:*

$$\lim_{\|x-y\|_X \to 0} \|Ax - Ay\|_Y = 0.$$

Example 102 *Define the operator $A : C[0,1] \to C[0,1]$ as*

$$Ax = \int_0^t x(s)\, ds.$$

We have

$$\begin{aligned}
\|Ax\| &= \sup_{t\in[0,1]} |Ax(t)| \\
&= \sup_{t\in[0,1]} \left| \int_0^t x(s)\, ds \right| \\
&\leq \sup_{t\in[0,1]} \int_0^t |x(s)|\, ds \\
&\leq \sup_{t\in[0,1]} \int_0^1 \|x\|\, ds \\
&= \|x\| \sup_{t\in[0,1]} \int_0^t ds \\
&= \|x\| \sup_{t\in[0,1]} t \\
&= \|x\|.
\end{aligned}$$

Thus,

$$\|Ax\| \leq \|x\|,$$

which shows that A is a bounded operator.

Example 103 *Define $A : C[0,1] \to \mathbb{R}$*

$$Ax = \int_0^1 x(s)\, ds$$

$$\begin{aligned}
\|Ax\| &= |Ax| \\
&= \left| \int_0^1 x(s)\, ds \right| \\
&\leq \int_0^1 |x(s)|\, ds \\
&\leq \int_0^1 \|x\|\, ds \\
&= \|x\|
\end{aligned}$$

Then A is bounded operator.

Example 104 *Define $T : X \to \mathbb{R}$ such that $Tx = \|x\|$. We want to show that T is a bounded operator. We have*

$$\begin{aligned}\|Tx\| &= |Tx| \\ &= |\|x\|| \\ &= \|x\| .\end{aligned}$$

Then, T is a bounded operator. Also, $\forall x, y \in X$:

$$\begin{aligned}\|Tx - Ty\| &= |Tx - Ty| \\ &= |\|x\| - \|y\|| \\ &\leq \|x - y\| ,\end{aligned}$$

leading to

$$\lim_{\|x-y\|\to 0} \|Tx - Ty\| = 0,$$

which means that T is continuous.

Example 105 *Let us show that the differential operator is not bounded. Consider the family of continuous functions on $[0, 1]$*

$$\{x_n\} = \{t^n\} \quad n = 1, 2, 3...$$

First, note that $\forall n \in \mathbb{N}$, the norm of x_n is defined by

$$\|x_n\| = \sup_{t\in[0,1]} |t^n| = 1.$$

We have

$$\begin{aligned}\left\|\frac{d}{dx}x_n\right\| &= \left\|\frac{d}{dx}t^n\right\| \\ &= \left\|nt^{n-1}\right\| \\ &= n\left\|t^{n-1}\right\| \\ &= n.\end{aligned}$$

Note, as $n \to \infty$:

$$\|x_n\| \to 1$$

but

$$\left\|\frac{d}{dx}x_n\right\| = n \to \infty.$$

Therefore, there does not exist $C > 0$ such that $\left\|\frac{d}{dx}x_n\right\| \leq C\|x_n\|$.

Theorem 106 *A linear operator is bounded if and only if it is continuous.*

Proof. If T is a linear operator, then

$$\begin{aligned} T(0) &= T(x-x) \\ &= Tx - Tx \\ &= 0. \end{aligned}$$

Now, let us divide the proof of the equivalence into two main parts:

First, if T is bounded then $\forall x, y \in X : \exists C > 0$. We have

$$\|Tx - Ty\| = \|T(x-y)\| \leq \|x-y\|,$$

and thus

$$\lim_{\|x-y\| \to 0} \|Tx - Ty\| = 0.$$

Hence, T is continuous. This proves the forward implication.

Second, assume T is not bounded. It follows that there exists a sequence $\{x_n\}$ such that

$$\forall n \in \mathbb{N} : n \|x_n\| \leq \|Tx_n\|.$$

Define a new sequence $\{x_n^*\}$ as

$$x_n^* = \frac{x_n}{n \|x_n\|},$$

for which

$$\begin{aligned} \|x_n^* - 0\| &= \|x_n^*\| \\ &= \left\| \frac{x_n}{n \|x_n\|} \right\| \\ &= \frac{\|x_n\|}{n \|x_n\|} \\ &= \frac{1}{n}, \end{aligned}$$

and

$$\begin{aligned} \|Tx_n^* - T0\| &= \|Tx_n^*\| \\ &= \left\| T \frac{x_n}{n \|x_n\|} \right\| \\ &= \frac{\|Tx_n\|}{n \|x_n\|} \geq 1. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|x_n^* - 0\| = 0,$$

but

$$\lim_{n\to\infty} \|Tx_n^* - T0\| \geq 1,$$

which implies that T is not continuous. ■

Remark 107 *The differential operator is linear unbounded, and thus it is not continuous.*

Example 108 *Define the operator $T : \ell_\infty \to \ell_\infty$ by $Tx = z$ where $x = (x_n)$, $z = (z_n)$, and*

$$z_n = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Let us show that T is linear, bounded and continuous.
First, we have

$$\begin{aligned} T(\alpha x + \beta y) &= \left(\alpha x_1 + \beta y_1, \frac{\alpha(x_1 + x_2) + \beta(y_1 + y_2)}{2}, ...\right) \\ &= \left(\alpha x_1, \frac{\alpha(x_1 + x_2)}{2}, ...\right) + \left(\beta y_1, \frac{\beta(y_1 + y_2)}{2}, ...\right) \\ &= \alpha\left(x_1, \frac{x_1 + x_2}{2}, ...\right) + \beta\left(y_1, \frac{y_1 + y_2}{2}, ...\right) \\ &= \alpha T(x) + \beta T(y), \end{aligned}$$

which implies that T is a linear operator.
Second,

$$\|Tx\|_{\ell_\infty} = \sup\left\{|x_1|, \frac{|x_1 + x_2|}{2}, ...\right\}.$$

It follows that $\forall n \in \mathbb{N}$:

$$\begin{aligned} \left|\frac{\sum_{i=1}^{n} x_i}{n}\right| &\leq \frac{\sum_{i=1}^{n} |x_i|}{n} \\ &\leq \frac{\sum_{i=1}^{n} \|x\|}{n} = \|x\|, \end{aligned}$$

and thus

$$\|Tx\|_{\ell_\infty} \leq \|x\|,$$

which means that T is bounded . Now, since T is linear and bounded, then it is continuous.

4.1 Fundamontal Theorems

Theorem 109 (Riezs repesentation theorem) *For every bounded linear functional f defined on a Hilbert space H, i.e. $f : H \to \mathbb{R}$, there exists a unique $z \in H$ such that*

$$f(x) = \langle x, z \rangle, \ \forall x \in H. \tag{4.1}$$

Remark 110 *The functional f defined by (4.1) is bounded and linear since*

$$\begin{aligned} f(\alpha x + \beta y) &= \langle \alpha x + \beta y, z \rangle \\ &= \alpha \langle x, z \rangle + \beta \langle y, z \rangle \\ &= \alpha f(x) + \beta f(y), \end{aligned}$$

and

$$\begin{aligned} \|f(x)\| &= |f(x)| \\ &= |\langle x, z \rangle| \leq \|x\| \|z\|, \end{aligned}$$

leadings to

$$\|f(x)\| \leq c \|x\|; c = \|z\|.$$

Hence, as f is bounded and linear, it is continuous.

Theorem 111 (Hahn-Banach theorem) *Assume G is a subspace of the normed space E. For every linear functional f on G, there exists a linear functional F defined on E such that*

$$f(x) = F(x), \ \forall x \in G.$$

Remark 112 *The functional F is called an extention of f.*

Definition 113 (Dual or conjugate space) *The dual space of the normed space E consists of all bounded linear functionals defined on E*

$$f : E \to \mathbb{R}.$$

This space is denoted by E^*.

Example 114 *For* $E = C[0,1]$ *and* $\|x\| = \sup |x(t)|$, *we define* $f : C[0,1] \to \mathbb{R}$ *by*

$$f(x) = \int_0^1 x(t)\,dt.$$

Since f *is a linear and bounded operator, then* $f \in C^*[0,1]$.

Remark 115 *If* E *is a Hilbert space, then all functionals forming the dual of the Hilbert space are of the form:*

$$f \in H^*,\ \forall x \in H,\ f(x) = \langle x, z \rangle,\ z \in H.$$

Theorem 116 *The dual space* E^* *with the norm*

$$\|f\|_{E^*} = \sup_{x \neq 0} \frac{|f(x)|_{\mathbb{R}}}{\|x\|_E}$$

is a Banach space.

Proposition 117 *For every* $f \in E^*$, *we have*

$$|f(x)| \leq \|f\| \cdot \|x\|,\ \forall x \in E. \tag{*}$$

Proof. We consider two separate cases:

The first is where $x = 0$, in which case (**??**) is true as $f(0) = 0$ and $\|0\| = 0$.

The second case is where $x \neq 0$, which yields

$$\frac{|f(x)|}{\|x\|_E} \leq \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_E} = \|f\|_{E^*}.$$

Therefore,

$$|f(x)| \leq \|f\|_{E^*} \cdot \|x\|_E.$$

■

Theorem 118 *The dual space of* $L_p[a,b]$ *is isomorphic to* $L_q[a,b]$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $p \in [1,\infty)$ *in the following sence:*
For each $f \in L_p^*[a,b]$, *there exists a corresponding* $g \in L_q[a,b]$ *such that*

$$f(x) = \int_a^b g(t)\,x(t)\,dt.$$

We write

$$L_p^*[a,b] \simeq L_q[a,b].$$

Example 119 *The following examples follow from the previous theorem*

$$\begin{aligned} L_2^*[a,b] &\simeq L_2[a,b], \\ L_2^{**}[a,b] &\simeq L_2[a,b], \\ L_3^*[a,b] &\simeq L_{\frac{3}{2}}[a,b] \ \left(\frac{1}{3}+\frac{1}{3/2}=1\right), \\ L_1^*[a,b] &\simeq L_\infty[a,b] \ \left(\frac{1}{1}+\frac{1}{\infty}=1\right). \end{aligned}$$

Example 120 *Considering that $p \in [1,+\infty)$ and$\frac{1}{q}+\frac{1}{p}=1$, we have*

$$\ell_p^* \simeq \ell_q,$$

and

$$\ell_5^* \simeq \ell_{\frac{5}{4}} \quad \left(\frac{1}{5}+\frac{1}{5/4}=1\right).$$

Exercise 121 *Show that $C_0^* \simeq \ell_1$ and $\ell_\infty^* \subset \ell_1$.*

Definition 122 *The normed space E is said to be reflexive if the second dual E^{**} is isomorphic to E; that is*

$$E^{**} = (E^*)^* \simeq E.$$

Lemma 123 *The following statements hold:*
(1) The spaces $L_p[a,b]$ and ℓ_p for $p \in (1,+\infty)$ are reflexive, i.e.

$$L_p^{**}[a,b] \simeq L_p[a,b],$$

and

$$\ell_p^{**} \simeq \ell_p.$$

(2) All Hilbert spaces are reflexive.
(3) The space C_0 is not reflexive as

$$\begin{aligned} C_0^{**} &= (C_0^*)^* \\ &= (\ell_1)^* \simeq \ell_\infty. \end{aligned}$$

Definition 124 (Weak and Strong Convergence) *Let $\{x_n\}$ be a sequence in the normed space $(E, \|\cdot\|)$:*
(1) We say that $\{x_n\}$ converges ***strongly*** *to x_0 if*

$$\lim_{n\to\infty} \|x_n - x_0\|_E = 0.$$

We write $x_n \to x_0$.
(2) We say that $\{x_n\}$ convergs ***weakly*** *to x_0 if for every $f \in E^*$,*

$$\lim_{n\to\infty} |f(x_n) - f(x_0)| = 0.$$

We write $x_n \rightharpoonup x_0$.

Remark 125 *Strong convergence implies weak convergence. Consider the following estimate*

$$|f(x_n) - f(x_0)| = |f(x_n - x_0)| \leq \|f\| \cdot \|x_n - x_0\|.$$

If the right hand side goes to zero, i.e.

$$\lim_{n\to\infty} \|x_n - x_0\|_E = 0,$$

then the right hand side does as well

$$\lim_{n\to\infty} |f(x_n) - f(x_0)| = 0.$$

Remark 126 *There are weakly converging sequences that do not converge strongly.*

Example 127 *Consider the Hilbert space ℓ_2 and the sequence $\{x_n\}$ where*

$$\begin{aligned} x_1 &= (1, 0, 0, 0, ...), \\ x_2 &= (0, 1, 0, 0, ...), \\ x_3 &= (0, 0, 1, 0, ...), \\ &\vdots \\ x_n &= (0, 0, ..., 0, 1, 0, ...). \end{aligned}$$

Note that

$$\lim_{n\to\infty} x_n = (0, 0, 0, 0, ...) =: x_0,$$

which yields

$$\begin{aligned}\|x_n - x_0\| &= \|(0,0,...,0,1,0,.....)\| \\ &= \|x_n\| \\ &= 1.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \|x_n - 0\| = 1 \neq 0.$$

Hence, this sequence is not strongly convergent. However, since ℓ_2 is a Hilbert space, then for any $f \in \ell_2$, there exists $a = \{a_n\} \in \ell_2$ such that $f(x) = \langle x, a\rangle$, $x \in \ell_2$, then

$$\begin{aligned}f(x_n) &= \langle x_n, a\rangle \\ &= a_n.\end{aligned}$$

Since

$$a = \{a_n\} \in \ell_2 \sum_{n=1}^{\infty} a_n^2 < \infty,$$

we have

$$\lim_{n\to\infty} |a_n| = 0.$$

This along with the fact that

$$\begin{aligned}|f(x_n) - f(0)| &= |f(x_n)| \\ &= |a_n|,\end{aligned}$$

leads to

$$\lim_{n\to\infty} |f(x_n) - f(0)| = 0,$$

then

$$x_n \rightharpoonup x_0 = 0.$$

Definition 128 (Adjoint Operator) *Let $(X, \langle\cdot,\cdot\rangle)$ be a an inner product space. The adjoint operator T^* of the operator $T : X \to X$ satisfies*

$$\forall x, y \in X : \langle Tx, y\rangle = \langle x, T^* y\rangle.$$

Example 129 *Define the operator $T : L_2[0,\infty] \to L_2[0,\infty]$ by*

$$Tx(t) = x\left(\frac{t}{5}\right) t \in [0,\infty).$$

We want to show that T^* can be defined as

$$T^*x(t) = 5x(5t).$$

We have

$$\begin{aligned} \langle Tx, y\rangle &= \int_0^\infty Tx(t)\, y(t)\, dt \\ &= \int_0^\infty x\left(\frac{t}{5}\right) y(t)\, dt, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \langle x, T^*y\rangle &= \int_0^\infty x(t)\, T^*y(t)\, dt \\ &= \int_0^\infty x(t) \cdot 5x(5t)\, dt. \end{aligned}$$

We can integrate by substitution. Let

$$\begin{cases} u = 5t, \quad du = 5dt \\ t = 0 \to u = 0 \\ t = \infty \to u = \infty. \end{cases}$$

Substituting yields

$$\langle x, T^*y\rangle = \int_0^\infty x\left(\frac{u}{5}\right) y(u)\, du. \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

Lemma 130 *If T^* is the adjoint operator of T, then*
(1) $\forall x, y \in X : \langle T^*x, y\rangle = \langle x, Ty\rangle$,
(2) $T^{**} = T$, and
(3) $\forall \alpha \in \mathbb{R} : (\alpha T)^* = \alpha T^*$.

Proof. First, for the field of real numbers $\mathbb{R}$, we have:
(1) $\forall x, y \in X$:

$$\begin{aligned} \langle T^* x, y \rangle &= \langle y, T^* x \rangle \\ &= \langle y, T^* x \rangle \\ &= \langle Ty, x \rangle \\ &= \langle x, Ty \rangle . \end{aligned}$$

Second, for the field of complex number $\mathbb{C}$, we have:
(1) We have

$$\begin{aligned} \langle T^* x, y \rangle &= \overline{\langle y, T^* x \rangle} \\ &= \overline{\langle y, T^* x \rangle} \\ &= \langle Ty, x \rangle \\ &= \langle x, Ty \rangle . \end{aligned}$$

(2) For the second property, we have

$$\begin{aligned} \langle T^{**} x, y \rangle &= \langle x, T^* y \rangle \\ &= \langle Tx, y \rangle , \end{aligned}$$

leading to

$$\forall x, y \in X : \ \langle T^{**} x, y \rangle = \langle Tx, y \rangle .$$

Thus,

$$T^{**} = T.$$

(3) For every $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} \langle (\alpha T)^* x, y \rangle &= \langle x, \alpha T y \rangle \\ &= \alpha \langle x, Ty \rangle \\ &= \alpha \langle T^* x, y \rangle \\ &= \langle \alpha T^* x, y \rangle . \end{aligned}$$

Therefore,

$$(\alpha T)^* = \alpha T^*.$$

Note that if $\alpha \in \mathbb{C}$,

$$(\alpha T)^* = \overline{\alpha} T^*.$$

■

Definition 131 *An operator T is said to be **self-adjoint** if $T = T^{*}$; that is $\forall x, y \in X : \langle Tx, y\rangle = \langle x, Ty\rangle$.*

Definition 132 *An operator T is said to be **unitary** if $T^{*} = T^{-1}$; that is $\forall x, y \in X : \langle Tx, y\rangle = \langle x, T^{-1}y\rangle$.*

Example 133 *Let $T : L_2[a, b] \to L_2[a, b]$ where*

$$Tx(t) = tx(t), t \in [a, b].$$

We will prove that T is self-adjoint. We have

$$\begin{aligned} \langle Tx, y\rangle &= \int_a^b Tx(t)\, y(t)\, dt \\ &= \int_a^b tx(t)\, y(t)\, dt, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \langle x, Ty\rangle &= \int_a^b x(t)\, Ty(t)\, dt \\ &= \int_a^b x(t)\, ty(t)\, dt. \end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we obtain that T is self-adjoint.

Example 134 *Define the operator $T : L_2[0, 1] \to L_2[0, 1]$ by*

$$Tx(t) = x(1 - t)$$

Observe that

$$\begin{aligned} \langle Tx, y\rangle &= \int_0^1 Tx(t)\, y(t)\, dt \\ &= \int_0^1 x(1 - t)\, y(t)\, dt. \end{aligned}$$

We use the following change of variable

$$\begin{cases} u = 1 - t \to du = -dt \\ t = 0 \to u = 1, t = 1 \to u = 0. \end{cases}$$

Substitution yields

$$\begin{aligned}\langle Tx, y\rangle &= -\int_1^0 x(u)\,y(1-u)\,du \\ &= \int_0^1 x(u)\,y(1-u)\,du \\ &= \int_0^1 x(u)\,Ty(u)\,du \\ &= \langle x, Ty\rangle\,.\end{aligned}$$

Hence, T is self-adjoint.

Lemma 135 *Let T and S be two operators defined on the inner product space $(X, \langle\cdot,\cdot\rangle)$, then*
(1) TT^* is self-adjoint, and
(2) $(ST)^* = T^*S^*$.

Proof. For the first property, we have

$$\langle TT^*x, y\rangle = \langle T^*x, T^*y\rangle\,,$$

leading to

$$\langle x, TT^*y\rangle = \langle T^*x, T^*y\rangle\,,$$

which implies that TT^* is self-adjoint.

For the second property,

$$\begin{aligned}\langle (ST)\,x, y\rangle &= \langle Tx, S^*y\rangle \\ &= \langle x, T^*S^*y\rangle\,.\end{aligned}$$

This produces

$$\langle (ST)\,x, y\rangle = \langle x, (ST)^*\,y\rangle\,,$$

which means that $(ST)^* = T^*S^*$. ■

Lemma 136 *If T is a unitary operator then:*
(1) It preserves the length of the element x, i.e. $\|Tx\| = \|x\|$ ..
(2) It preserves of the angle, i.e. $\langle Tx, Ty\rangle = \langle x, y\rangle$.

Proof. For the length, we have

$$\begin{aligned} \|Tx\|^2 &= \langle Tx, Ty \rangle \\ &= \left\langle x, T^{-1}Tx \right\rangle \\ &= \langle x, x \rangle \\ &= \|x\|^2 , \end{aligned}$$

which leads to property (1).

For the angle, it is easy to see that

$$\begin{aligned} \langle Tx, Ty \rangle &= \left\langle x, T^{-1}Ty \right\rangle \\ &= \langle x, y \rangle . \end{aligned}$$

■